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On Constrained Minimizers for Schrödinger Equations with Hardy Term

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Abstract

In this article, our focus lies on a Schrödinger equation incorporating a Hardy term. To identify the global minimizers of the functional I under a mass constraint, we utilize the Hardy inequality. In addition, we reveal that every energy ground state is directly linked to the least action solution of the associated action functional. This finding affirmatively addresses the question of whether, under broad assumptions, The functional I is characterized by a mountain pass structure and satisfies the $(PS)_c$ condition. Subsequently, this guarantees the existence of a nontrivial critical point u for the energy functional I.

Subject Areas

Functional Analysis

Keywords

Schrödinger Equations, Hardy Term

1. Introduction

Let $N \geq 3$ and define a C^1 energy functional I mapping from $H^1(\mathbb{R}^N)$ to $\mathbb{R}$ as follows:

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 - \frac{\mu}{|x|^2}u^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} G(u)\,\mathrm{d}x, \tag{1.1}$$

where $g \in C(\mathbb{R},\mathbb{R})$ and $G(t) := \int_0^t g(\tau)\,\mathrm{d}\tau$.

We re-exlore the traditional problem of identifying conditions on $G \in C^1(\mathbb{R},\mathbb{R})$ that make the existence of global minimizers for I with a mass constraint.

$$S_m := \left\{u \in H^1(\mathbb{R}^N) \,|\, \|u\|_2^2 = m\right\}.$$

In this article, we delve into the minimization problem

$$E_m := \inf_{u \in S_m} I(u), \tag{1.2}$$

where $m > 0$ is prescribed.

By directly using the Lagrange multiplier method, Suppose $u \in S_m$ is a solution to (1.2), then a corresponding $\lambda = \lambda(u) \in \mathbb{R}$ exists such that

$$-\Delta u - \frac{\mu}{|x|^2} u = -\lambda u + g(u) \text{ in } H^1(\mathbb{R}^N). \tag{1.3}$$

A solution minimizing (1.2) is often referred to as an energy ground state, and the corresponding ground state energy is denoted by E_m. The problem about constrained minimization for the Schrödinger equation incorporating a Hardy term is a vibrant area of investigation within the realm of mathematical physics, involving multiple different research directions. The origins of research in this area can be traced back to at least 2004, with Dider Smets's work on nonlinear Schrödinger equations that include a Hardy potential and a critical Sobolev exponent, see [1]. This paper offers an in-depth examination of the gradient flow lines and critical points at infinity, establishing a crucial theoretical groundwork for follow-up studies.

In recent years, research in this field has continued to deepen, involving multiple different directions. As an example, a research paper focused on the normalized ground state solutions of the Sobolev critical Schrödinger equation with a Hardy term and combined nonlinearities in [2]. Moreover, studies have delved into the constrained minimization issue concerning the nonlinear Schrödinger equation that incorporates the Anderson Hamiltonian. This is pertinent to the transmission dynamics of quantum particles through disordered media in the field of physics, see [3].

In 2017, researchers also studied the minimization problem of the Schrödinger-Poisson-Slater equations, which are important in describing the interactions between particles in many-body quantum systems with unbounded potential in [4]. In 2023, additional explorations were conducted into the normalized ground state solutions for the Hardy-Littlewood-Sobolev critical Schrödinger equation involving double Choquard-type nonlinear terms, see [5].

These studies indicate that the constrained minimization problem for the Schrödinger equation with Hardy term is not only mathematically challenging but also widely utilized across different branches of physics, especially in quantum mechanics and condensed matter physics. Over time, research in this area continues to expand, involving more physical phenomena and mathematical problems. In physics, the Schrödinger equation with Hardy term is primarily applied to describe the quantum mechanical systems of particles. The Hardy term is usually related to the geometric properties of space, and it introduces a potential energy that depends on the particle's position. This is particularly useful when dealing with problems that have special spatial symmetries or different physical properties in certain directions.

The applications of the Schrödinger equation with Hardy term in physical research are diverse, and recent studies have mainly focused on the following areas. Hardy's paradox provides a theoretical framework that highlights the tension between quantum mechanics and local realism. Such paradoxes can potentially contribute to the development of quantum information science, particularly in quantum computing and quantum communication. As an example, a new framework addressing multi-particle Hardy's paradox has been presented in [6], which may have significant implications for quantum information processing. The Hardy term also has a profound influence on the study of quantum entanglement and nonlocality. Through Hardy's inequality, it is possible to detect whether more quantum states exhibit nonlocality, which is beneficial for the experimental verification of quantum entanglement and the development of quantum communication protocols. In mathematical physics, the Schrödinger equation with a Hardy term is often associated with some challenging mathematical problems, such as critical point theory, variational methods, and nonlinear analysis. The works in References [7] and [5] focused on studying the normalized ground state solutions for the Schrödinger equation with a Hardy term and critical Sobolev exponent.

The following assumptions on $g \in C(\mathbb{R},\mathbb{R})$ will be required.

(g_1) $\lim\limits_{t\to 0} g(t)t^{-1} = 0$.

(g_2) $\limsup\limits_{|t|\to\infty} \dfrac{|g(t)|}{|t|^{\frac{N+2}{N-2}}} < \infty$.

(g_3) $G(\zeta) > 0$ exists $\zeta \neq 0$.

(g_4) There exists $q \in (2,2^*)$ and $C > 0$ such that $|g(t)| \le C\left(|t| + |t|^{q-1}\right)$ for every $t \in \mathbb{R}$.

(g_5) There exists $C_1, C_2 > 0$ and $\sigma \in (0,4)$, such that $g(t)t - 4G(t) \ge C_1|t|^{2+\sigma} - C_2 t^2$ for every $t \in \mathbb{R}$.

(g_6) $\lim\limits_{t\to\infty} G(t)t^{-4} = \infty$, there $G(t) = \int_0^t g(u)\mathrm{d}u$.

Theorem 1.1 Assume $N \ge 3$ and $g \in C(\mathbb{R},\mathbb{R})$ complies with conditions (g_1) to (g_3). Thus

$$E_m := \inf_{u\in S_m} I(u) - \infty$$

and the function $m \mapsto E_m$ is continuous and non-increasing. Besides,

i) There is a value $m^* \in [0,\infty)$ obtain

$$E_m = 0 \text{ if } 0 < m \le m^*, \quad E_m < 0 \text{ when } m > m^*;$$

ii) For $m > m^*$, the global minimum E_m is reached, and consequently, (1) has an energy ground state $v \in S_m$ with $I(v) = E_m < 0$.

iii) when $0 < m < m^*$, the value $E_m = 0$ is not obtained;

iv) $m^* = 0$ when

$$\lim_{|t|\to 0} G(t) \Big/ |t|^{2+\frac{4}{N}} = +\infty, \tag{1.4}$$

and $m^* > 0$ when

$$\limsup_{|t|\to 0} G(t) \Big/ |t|^{2+\frac{4}{N}} < +\infty. \tag{1.5}$$

Remark 1.2 i) As further explained in Remark 3.3 and demonstrated in the proof of Theorem 1.1 (ii), and if $m > m^*$, it is shown that any minimizing sequence for (1.2), after selecting a subsequence and modulo translations in $\mathbb{R}^N$, converges strongly.

ii) For $0 < m < m^*$, Theorem 1.1 (iii) gets that the global minimum $E_m = 0$ is not reached. Nevertheless, this does not exclude the possibility that the constrained functional $I_{|S_m}$ may have critical points with positive energies, as noted in the related work [8].

iii) In the context where $m^* > 0$, the exploration of the existence and nonexistence of global minimizers for $E_{m^*} = 0$ is a matter. For this surpasses the limits of the current paper, we will not examine more general conditions on f that ensure either existence or nonexistence. Instead, we guide interested readers to [8] and [9] for some results related to existence.

iv)To let the exposition more straightforward, we define the notation.

$$m \succeq_f m^*$$

To prove that $m \geq m^*$ if $m^* > 0$ and $E_{m^*} = 0$ is obtained, and $m > m^*$ otherwise. It is important to find that when $m \succeq_f m^*$ and for any minimizer $v \in S_m$ of (1.2), the associated Lagrange multiplier $\lambda = \lambda(v)$ is positive. Indeed, the Pohozaev identity associated with (1.3), as described in [10],

$$P(u) := N-2/2N \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} \frac{\mu}{|x|^2} u^2 \mathrm{d}x + \frac{1}{2}\lambda \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} G(u) \mathrm{d}x.$$

Given that $E_m = I(u) \leq 0$, it follows that

$$0 \geq= I(u) - P(u) = I(u) = 1/N \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - 1/2\, \lambda m$$

which implies that $\lambda > 0$.

Theorem 1.3 Let $N \geq 3$ and $g \in C(\mathbb{R}, \mathbb{R})$ fulfills conditions (g_1) through (g_3); thus the next results get:

i) The function u defined on S_m with respect to I represents a ground state of (1.1) when λ is equal to $\lambda(u)$ constitutes a solution of (1.1) having $\lambda = \lambda(u)$, meaning that

$$C_\lambda = E_m + \frac{\lambda}{2} m =: \inf\left\{ J_\lambda(v) \mid J_\lambda'(v) = 0 \right\}. \tag{1.6}$$

where $v \in H^1(\mathbb{R}^N) \setminus \{0\}$. For next detail, (1.6) the least action is C_λ the value, where action function with C^1 $J_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ is defined:

$$J_\lambda(v) = I(v) + \frac{\lambda}{2}\int_{\mathbb{R}^N} |v|^2 \mathrm{d}x. \tag{1.7}$$

ii) For any $\lambda \in \left\{ \lambda(u) : u \in S_m \text{ and } u \text{ minimizes } I \text{ on } S_m \right\}$, any ground state $w \in H^1(\mathbb{R}^N)$ of (1.3) minimizes I on S_m, meaning $w \in S_m$ and $I(w) = E_m$.

As we all know, the simplest and one of the most minimax theorems is the mountain pass theorem. Now let's recall it.

Theorem 1.4 ([11]) Suppose E be a Hilbert space, $I \in C^1(E, \mathbb{R})$, and $e \in E$ where $\|e\| > p$ make

$$b := \inf_{\substack{v \in E \\ \|v\| = p}} I(v) \geq I(0) > I(e).$$

Thus, there exists $\{v_n\} \in E$ obtain

$$\lim_{n \to \infty} I(v_n) = c, \quad \lim_{n \to \infty} \|I'(v_n)\| = 0$$

with

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

$$\Gamma := \{\gamma \in C([0,1], E) : \gamma(1) = e, \gamma(0) = 0\}.$$

and when the function I satisfies the $(PS)_c$ condition, c is the solution of I.

Theorem 1.5 Let that the function g make conditions (g_1) to (g_6) is hold, thus the energy function I obtained a nontrivial solution.

The organization of this paper is as follows: the section 2 introduces the necessary notation and presents a preliminary lemma. The section 3 is in order to prove Theorem 1.1. the section 4 focuses on demonstrating Lemma 4.1 and Theorem 1.3. Lastly, Section 5 is allocated to prove Theorem 1.4 and Theorem 1.5.

2. Preliminaries

Throughout this paper, for a given function $u \in H^1(\mathbb{R}^N)$ and for any real number s, we introduce the scaling function defined as

$$s \lozenge u := \mathrm{e}^{Ns/2} u(\mathrm{e}^s \cdot)$$

which remains in $H^1(\mathbb{R}^N)$ and maintains the L^2 norm as s varies over $\mathbb{R}$. In what follows, The Hardy inequality (as stated in [2]), which will be frequently used all over this paper, is recalled by us.

Lemma 2.1 (Hardy inequality [12]) For any $u \in C_0^\infty(\mathbb{R}^N)$ and $N \geq 3$, we have a sharp number $\mu' = (N - 2/2)^2$, $N \geq 3$, make

$$\int_{\mathbb{R}^N} 1/|x|^2 u^2 \mathrm{d}x \leq \frac{1}{\mu'} \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x, \quad 0 \leq \mu < \mu'$$

Given that functions in $C_0^\infty(\mathbb{R}^N)$ are dense in $H^1(\mathbb{R}^N)$, the aforementioned inequality is also valid in $H^1(\mathbb{R}^N)$.

In what follows, we denote that $C_0^\infty(\mathbb{R}^N)$ is $D^{1,2}(\mathbb{R}^N)$ the completion, where the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{1}{|x|^2} u^2 \right) \mathrm{d}x \right)^{1/2}, \text{ for all } 0 \leq \mu < \mu',$$

where $u \in H^1(\mathbb{R}^N)$.

Then, this norm $\|u\|$ and the usual norm $\left(\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$ is equivalent.

3. Existence

The section is dedicated to proving Theorem 1.1, with a particular idea to proving the existence as well as the absence of minimizers for (1.2) within a suitable mass range $m>0$. For a prerequisite, Here, we introduce the following lemma, for which the proof follows conventional methods.

Lemma 3.1 Let that $N\geq 3$ and $g\in C(\mathbb{R},\mathbb{R})$ satisfies (g_1) and (g_2). Thus the next statements is achieved.

i) When every bounded sequence $\{v_n\}$ in $H^1(\mathbb{R}^N)$,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}G(v)\,\mathrm{d}x=0$$

if $\lim_{n\to\infty}\|v_n\|_{L^\infty(\mathbb{R}^N)}=0$, and

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}G(v)\,\mathrm{d}x\leq 0$$

if $\lim_{n\to\infty}\|v_n\|_{L^{2+4/N}(\mathbb{R}^N)}=0$.

ii) A positive number $C=C(g,N,m)$ makes

$$\begin{aligned}I(v)&=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2-\frac{\mu}{|x|^2}v^2\right)\mathrm{d}x-\int_{\mathbb{R}^N}G(v)\,\mathrm{d}x\\&\geq\frac{1}{2}\left(1-\frac{\mu}{\bar{\mu}}\right)|\nabla v|_2^2-C(g,N,m),\end{aligned}$$

for any $v\in H^1(\mathbb{R}^N)$ and $|v|_2^2\leq m$. Specifically, the function I is coercive on S_m.

To delve deeper, we revisit the global minimum

$$E_m:=\inf I(v)$$

where $v\in S_m$, subsequently conduct a thorough analysis of its fundamental characteristics.

Lemma 3.2 Let $N\geq 3$ and $g\in C(\mathbb{R},\mathbb{R})$ fulfills conditions (g_1) through (g_3). The next assertions are valid.

i) for all $m\geq 0$, $-\infty<E_m\leq 0$.

ii) There exists a value $m_0\geq 0$, $E_m<0$ with every $m>m_0$.

iii) If (1.4) is satisfied, then $E_m\leq 0$ for all $m\geq 0$. Conversely, if (1.5) is achieved, thus $E_m=0$ for small enough $m\geq 0$.

iv) When every $m>m'>0$, it is achieved that

$$E_m\leq\frac{m}{m'}E_{m'}\tag{3.1}$$

When $E_{m'}$ is attained, thus it is the strict inequality.

v) The function that assigns $m\mapsto E_m$ is monotonically non-increasing and exhibits continuity.

Proof: Due to lemma 3.1 can be proved similarly as that of Lemma 2.1 in [13], so it is omitted by us.

i) Due to Lemma 3.1 (ii), I has a lower bound on S_m, we get $E_m > -\infty$. For every $u \in L^{\infty}\left(\mathbb{R}^N\right) \cap S_m$, we get $\lim\limits_{s \to -\infty} \left|\nabla\left(s \diamond v\right)\right|_2 = 0$ and $\lim\limits_{s \to -\infty} \left|\nabla\left(s \diamond v\right)\right|_{L^{\infty}\left(\mathbb{R}^N\right)} = 0$. Referring to Lemma 3.1(i),

$$
\begin{aligned}
E_m &\leq \lim_{s \to -\infty} I\left(s \diamond v\right) \\
&= \frac{1}{2}\int_{\mathbb{R}^N}\left(\left|\nabla s \diamond v\right|^2 - \frac{\mu}{|x|^2}\left|s \diamond v\right|^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} G\left(s \diamond v\right)\mathrm{d}x \\
&= \lim_{s \to -\infty} -\frac{\mu}{2}\int_{\mathbb{R}^N}\frac{\mathrm{e}^{2s}|v|^2}{|x|^2}\mathrm{d}x = 0.
\end{aligned}
$$

ii) From $\left(g_3\right)$ and [10], we get $v \in H^1\left(\mathbb{R}^N\right)$, we know $\int_{\mathbb{R}^N} G\left(v\right)\mathrm{d}x > 0$. When all $m \geq 0$, let $v_m := v\left(m^{-1/N}|v|_2^{2/N}x\right) \in S_m$. From

$$
\begin{aligned}
I\left(v_m\right) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(\left|\nabla v_m\right|^2 - \frac{\mu}{|x|^2}v_m^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} G\left(v_m\right)\mathrm{d}x \\
&= \frac{m^{\frac{N-2}{N}}}{2|v|_2^{2(N-2)/N}}\int_{\mathbb{R}^N}\left(|\nabla v|^2 - \mu\frac{v^2}{|x|^2}\right)\mathrm{d}x - \frac{m}{|v|_2^2}\int_{\mathbb{R}^N} G\left(v\right)\mathrm{d}x \\
&= Am^{N-2/N} - Bm =: f\left(m\right),
\end{aligned}
$$

where $A = \dfrac{\|v\|}{2|u|_2^{2(N-2)/N}}$, $B = \dfrac{1}{|v|_2^2}\int_{\mathbb{R}^N} G\left(v\right)\mathrm{d}x$.

Consequently, we have $E_m \leq I\left(v_m\right) = f\left(m\right) < 0$, with every large enough $m > 0$.

iii) If (1.4) is achieved, we select $v \in L^{\infty}\left(\mathbb{R}^N\right) \cap S_m$. When

$$
H := \int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x \Big/ \int_{\mathbb{R}^N}|v|^{2+4/N}\,\mathrm{d}x > 0
$$

by (1.4), we get $\delta > 0$ making $G\left(t\right) \geq H|t|^{2+4/N}$ for any $|t| \leq \delta$. Given that $\|s \diamond v\|_{L^{\infty}\left(\mathbb{R}^N\right)} \leq \delta$ for some $s < 0$, it follows clearly that

$$
\begin{aligned}
E_m &\leq I\left(s \diamond v\right) \\
&= \frac{1}{2}\int_{\mathbb{R}^N}\left(\left|\nabla s \diamond v\right|^2 - \frac{\mu}{|x|^2}\left|s \diamond v\right|^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} F\left(s \diamond v\right)\mathrm{d}x \\
&\leq \frac{1}{2}\mathrm{e}^{2s}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x - H\mathrm{e}^{2s}\int_{\mathbb{R}^N}|v|^{2+\frac{4}{N}}\,\mathrm{d}x - \frac{1}{2}\mu\mathrm{e}^{2s}\int_{\mathbb{R}^N}\frac{\left|v\left(x\right)\right|^2}{|x|^2}\mathrm{d}x \\
&= -\frac{1}{2}\mathrm{e}^{2s}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x - \frac{1}{2}\mathrm{e}^{2s}\mu\int_{\mathbb{R}^N}\frac{\left|v\left(x\right)\right|^2}{|x|^2}\mathrm{d}x > 0.
\end{aligned}
$$

When (1.5) holds, there exists a positive number C_g making $G\left(t\right) \leq C_g|t|^{2+4/N}$, with any $t \in \mathbb{R}$. During the Gagliardo-Nirenberg inequality, we all know

$$
\int_{\mathbb{R}^N} G\left(u\right)\mathrm{d}x \leq C_g C_N m^{2/N}|\nabla v|_2^2 \quad \text{for all } v \in S_m.
$$

For small enough $m > 0$, one has $C_g C_N m^{2/N} < \frac{1}{2}\left(1 - \frac{\mu}{\mu'}\right)$, Due to Lemma 2.1 such that

$$I(v) := \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 - \frac{\mu}{|x|^2}v^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} G(v)\,\mathrm{d}x$$
$$\geq \frac{1}{2}\left(1 - \frac{\mu}{\mu'}\right)|\nabla v|_2^2 - C_g C_N m^{2/N}|\nabla v|_2^2 > 0.$$

Therefore, $0 \leq E_m$. According to (i), we get that $E_m = 0$ when $m > 0$ small.

iv) Suppose $1 < b = m/m'$. When all $\varepsilon > 0$, we have a function $u \in S_{m'}$ and get $I(u) \leq \varepsilon + E_{m'}$. It is evident that $v \in S_m$, $v = u\left(t^{-1/N}x\right)$, then

$$\begin{aligned} E_m \leq I(v) &= 1/2\int_{\mathbb{R}^N}\left(\left|\nabla u\left(t^{-\frac{1}{N}}x\right)\right|^2 - \frac{\mu}{|x|^2}u\left(t^{-\frac{1}{N}}x\right)^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} G\left(u\left(t^{-\frac{1}{N}}x\right)\right)\mathrm{d}x \\ &= \frac{1}{2}t^{\frac{N-2}{N}}\left(1 - t^{\frac{2}{N}}\right)\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x + tI(u) - \mu t\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^2}\mathrm{d}x \\ &< tI(u) \\ &\leq \frac{m}{m'}\left(E_{m'} + \varepsilon\right). \end{aligned} \tag{3.2}$$

Since $\varepsilon > 0$ can be chosen arbitrarily, we can know that (3.1) the inequality is hold. When $E_{m'}$ is attained, for a few $u \in S_{m'}$, thus we can let $\varepsilon = 0$ with (3.2), such that the inequality is strict.

v) Due to (3.1) and Item (i), the function E_m is nonincreasing. For establishing E_m continuity, all $m > 0$ and at a $v \in S_1$, we define a new function.

$$\begin{aligned} \phi_v(m) &:= 1/m\, I\left(v\left(m^{-}\frac{1}{N}x\right)\right) \\ &= 1/2\, m^{-2/N}\int_{\mathbb{R}^N}\left(|\nabla v|^2 - \frac{\mu}{|x|^2}|u|^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} G(v)\,\mathrm{d}x \\ &= 1/2\, m^{2/N}\|v\|^2 - \int_{\mathbb{R}^N} G(v)\,\mathrm{d}x. \end{aligned}$$

we can obtain

$$\frac{E_m}{m} = \inf_{v \in S_1}\phi_v(m).$$

Given that $\phi_m(m)$ is concave about $m^{-\frac{2}{N}}$, we can learn that $\frac{E_m}{m}$ is continuous when $m > 0$, which in turn implies the continuity of E_m.

Demonstration of Theorem 1.1

We introduce the definition of

$$m^* := \inf\{m \mid E_m < 0\},$$

where m >0. From Lemma 3.2, it can be readily observed, if $0 \leq m^* < \infty$,

$$\text{if } m > m^* \ E_m < 0, \text{otherwise}, \ E_m = 0 \ \text{ when } 0 < m \le m^*; \tag{3.3}$$

Specifically, if (1.4) is satisfied, $m^* = 0$, and when (1.5) is held. $m^* > 0$. Let's initially demonstrate that $E_m = 0$ when $0 < m < m^*$ is unattainable. In fact, suppose for the sake of contradicting about $E_m = 0$ is attained when $m \in (0, m^*)$, thus we can get from Lemma tt (iv) that

$$0 = m^* / m \, E_m > E_{m^*}.$$

The results is contradicted since $E_{m^*} = 0$ from (3.3). Next, we ought to prove that E_m the global minimum is attained if $m > m^*$.

Given $m > m^*$ and learn every minimizing sequence $\{u_n\} \subset S_m$ about E_m. It is important that the bounded sequence $\{u_n\}$ in $H^1(\mathbb{R}^N)$ from Lemma 3.1 (ii). Consequently, suppose, having a subsequence, that $\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x$ and $\lim_{n\to\infty} \int_{\mathbb{R}^N} G(v_n) \mathrm{d}x$ exists. From (3.3), $E_m < 0$, it follows that the sequence $\{u_n\}$ does not vanish, that is to say

$$\lim_{n\to\infty} \left(\sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_n|^2 \, \mathrm{d}x \right) > 0. \tag{3.4}$$

In fact, if (3.4) were not valid, then according to Lions' Lemma [14] $u_n \to 0$ and consequently

$$0 \le \lim_{n\to\infty} \int_{\mathbb{R}^N} G(u_n) \mathrm{d}x$$

from Hardy inequality and Lemma 3.1 (i); Observing that
$I(u_n) + \int_{\mathbb{R}^N} G(u_n) \mathrm{d}x \ge 0$, it is contradicted that:

$$\begin{aligned} E_m &= \lim_{n\infty} I(v_n) < 0 \\ &= \lim_{n\to\infty} \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 - \frac{\mu}{|x|^2} v_n^2 \right) \mathrm{d}x - \int_{\mathbb{R}^N} G(v_n) \mathrm{d}x \\ &\ge \lim_{n\to\infty} \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, \mathrm{d}x - \frac{\mu}{2\mu'} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, \mathrm{d}x - \int_{\mathbb{R}^N} G(v_n) \mathrm{d}x \\ &\ge -\lim_{n\to\infty} \int_{\mathbb{R}^N} G(v_n) \mathrm{d}x \ge 0, \end{aligned}$$

where $0 \le \mu < \mu'$.

Given that the sequence $\{v_n\}$ does not vanish, we have one sequence $\{x_n\} \subset \mathbb{R}^N$ and a nontrivial element $u \in H^1(\mathbb{R}^N)$ making, up to a subsequence, $v_n(\cdot + x_n) \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and $v_n(\cdot + x_n) \to u$ a.e. on $\mathbb{R}^N$. Soppose $\overline{m} := \|u\|^2_{L^2(\mathbb{R}^N)}$, $0 < \overline{m} \le m$ and $h_n := v_n(\cdot + x_n) - u$. It is important that

$$\lim_{n\to\infty} |h|_2^2 = m - \overline{m} \tag{3.5}$$

and from employing the splitting result [15],

$$E_m = \lim_{n\to\infty} I(v_n) = \lim_{n\to\infty} I(u + h_n) = I(u) + \lim_{n\to\infty} I(h_n). \tag{3.6}$$

Now, we will demonstrate a claim below, which will subsequently complete the entire proof.

Claim. $\lim_{n\to\infty}|h_n|_2 = 0$. Specifically, it $m' = m$, from (3.5).

Let $t_n := |h_n|_2$ from every $n \in \mathbb{N}^+$. When $0 < \lim_{n\to\infty} t_n$, thus (3.5) obtain that $\overline{m} \in (0, m)$. Considering the definition about E_{t_n} and Lemma 3.2 (v), we derive that

$$\lim_{n\to\infty} I(h_n) \geq \lim_{n\to\infty} E_{t_n} = E_{m-\overline{m}}.$$

Fro (3.6) and (3.1), we get

$$E_m \geq I(u) + E_{m-\overline{m}} \geq E_{\overline{m}} + E_{m-\overline{m}} \geq \frac{m-\overline{m}}{m} E_m + \frac{\overline{m}}{m} E_m = E_m.$$

Thus, it must be that $I(u) = E_{\overline{m}}$, which indicates that $E_{\overline{m}}$ is attained at $v \in S_{\overline{m}}$. However, using (3.6) and (3.1) again, we find that it is contradicted:

$E_m \geq E_{\overline{m}} + E_{m-\overline{m}} > \overline{m}/m\, E_m + m - \overline{m}/m\, E_m = E_m$,

and thus the claim is established.

Summary. It is evident that $u \in S_m$ according to the claim above, and therefore $I(u) \geq E_m$. The claim, along with Lemma 3.1 (i), leads to (3.7)

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} G(h_n)\,\mathrm{d}x \leq 0. \tag{3.7}$$

By invoking the Hardy inequality, we additionally obtain

$$\begin{aligned}\lim_{n\to\infty} I(h_n) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla h_n|^2 - \frac{\mu}{|x|^2}|h_n|^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} G(h_n)\,\mathrm{d}x \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla h_n|^2 - \frac{\mu}{\mu'}|\nabla h_n|^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} G(h_n)\,\mathrm{d}x \\ &\geq -\int_{\mathbb{R}^N} G(h_n)\,\mathrm{d}x \geq 0.\end{aligned}$$

Consequently, leveraging (3.6), we deduce that $E_m \geq I(u)$, which implies when $u \in S_m$, the negative value $E_m < 0$ is attained.

Remark 3.3. We can deduce that $v_n(\cdot + x_n) \to u$ in $H^1(\mathbb{R}^N)$. Indeed, combining (3.6), (3.7) and the fact that $I(u) = E_m$, we can deduce that

$$\|\nabla h_n\|_{L^2(\mathbb{R}^N)} \to 0 \text{ as } n \to \infty.$$

Given that $\lim_{n\to\infty}\|h_n\|_{L^2(\mathbb{R}^N)} = 0$, the strong convergence follows.

4. Least Action Characterization

The section 4 is order to prove Theorem 1.3. As a prerequisite, we present the next proof, whose lemma is conventional.

Lemma 4.1 Let the function g fulfills (g_1)-(g_2) and $N \geq 3$. Due to every nontrivial critical point $h \in H^1(\mathbb{R}^N)$ of J_{μ_1}, any $\delta > 0$ and any $T > 0$, there is a number $Z = Z(h, \delta, T) > 0$ and a continuous path $\gamma : [0, Z] \to H^1(\mathbb{R}^N)$ that get

i) $\gamma(0) = 0$, $\max_{t\in[0,Z]} J_\mu(\gamma(t)) = J_\mu(h)$, $J_\mu(\gamma(Z)) < -1$;

ii) For some $\tau \in (0, Z)$, $\gamma(\tau) = h$, and

$$J_\lambda\left(\gamma(t)\right) < J_\lambda(h)$$

for each $t \in [0, Z]$, we have $\left\|\gamma(t) - h\right\|_{H^1(\mathbb{R}^N)} \geq \delta$;

iii) The function $m(t) := \left|\gamma(t)\right|_2^2$ is continuous and strictly increasing, with $M < m(Z)$.

Proof: When $N \geq 3$ and the fixed $h \in H^1\left(\mathbb{R}^N\right)$, we introduce

$$\gamma(t) := \begin{cases} 0 & \text{when } t = 0, \\ w(\cdot/t), & \text{when } t > 0. \end{cases} \tag{4.1}$$

One has

$$m(t) := \left|\gamma(t)\right|_2^2 = t^N \left|h\right|_2^2, \tag{4.2}$$

and by the Pohozaev identity, see [10].

$$P(u) := \frac{N-2}{2N}\int_{\mathbb{R}^N} \left|\nabla u\right|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} \frac{\mu}{|x|^2} u^2 \mathrm{d}x + \frac{1}{2}\lambda \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} G(u)\mathrm{d}x,$$

we note

$$\begin{aligned} J_\lambda\left(\gamma(t)\right) &= 1/2\int_{\mathbb{R}^N}\left(\left|\nabla\gamma(t)\right|^2 - \frac{\mu}{|x|^2}\gamma(t)^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} G\left(\gamma(t)\right)\mathrm{d}x + \frac{\lambda}{2}\int_{\mathbb{R}^N}\left|\gamma(t)\right|^2 \mathrm{d}x \\ &= 1/2\, t^{N-2}\int_{\mathbb{R}^N}\left(\left|\nabla h\right|^2 - \frac{\mu}{|x|^2}h^2\right)\mathrm{d}x - t^N\int_{\mathbb{R}^N} G(h)\mathrm{d}x + \frac{t^N}{2}\lambda\int_{\mathbb{R}^N}|h|^2 \mathrm{d}x \\ &= 1/2\left(t^{N-2} - \frac{N-2}{N}t^N\right)\int_{\mathbb{R}^N}\left|\nabla h\right|^2 \mathrm{d}x + \frac{1}{2}\left(t^N - t^{N-2}\right)\int_{\mathbb{R}^N}\frac{\mu}{|x|^2}h(x)^2 \mathrm{d}x \end{aligned}$$

with $0 \leq \mu < \mu'$, Suppose μ small enough. It is important that the function $J_\lambda\left(\gamma(t)\right)$ has a unique maximum at $t = 1$ and approaches $J_\lambda\left(\gamma(t)\right) \to -\infty$ as $t \to \infty$. Thus, for any $T > 0$ we can select a large enough number $Z = Z(h, T) > 0$ making the continuous path $\gamma : [0, Z] \to H^1\left(\mathbb{R}^N\right)$ meets Items (i) - (iii) of Lemma 4.1 about each $\delta > 0$.

When $N \geq 3$, We will observe that Lemma 4.1 the proof is quite straightforward. Finally, we depend on the existence of the Pohozaev identity with (1.3).

Demonstration of Theorem 1.3

For establishing Item (i), set $h \in H^1\left(\mathbb{R}^N\right)$ be any nontrivial critical point about J_λ, We should prove that

$$J_\lambda(h) \geq J_\lambda(u) = \frac{\lambda}{2}m + E_m.$$

Given a fixed $0 < \delta$ and supposed $T := m > 0$, consider the path $\gamma : [0, Z] \to H^1\left(\mathbb{R}^N\right)$ continuous as described in Lemma 4.1. According to Lemma 4.1 (i) and 4.1 (iii), there is a point $t_0 \in (0, T)$ for which

$$\left|\gamma\left(t_0\right)\right|_2^2=m.$$

Therefore,

$$\begin{aligned}J_\lambda(h)&=\max_{t\in[0,Z]}J_\lambda(\gamma(t))\geq J_\lambda\left(\gamma\left(t_0\right)\right)\\&=\frac{\lambda}{2}\int_{\mathbb{R}^N}\left|\gamma\left(t_0\right)\right|^2\mathrm{d}x+I\left(\gamma\left(t_0\right)\right)\\&\geq\frac{\lambda}{2}m+E_m.\end{aligned}$$

We now proceed to prove Item (ii). Given Item (i), any least action solution $h\in H^1\left(\mathbb{R}^N\right)$ of (1.3) fulfills

$$J_\lambda(h)=C_\lambda=\frac{\lambda}{2}m+E_m. \tag{4.3}$$

Suppose, for the sake of contradiction, that $\|w\|_{L^2\left(\mathbb{R}^N\right)}^2\neq m$. Thus, for

$$\delta:=\left|-|h|_2+\sqrt{m}\right|>0\ \text{ and }\ T:=m>0,$$

we have the path $\gamma:[0,Z]\to H^1\left(\mathbb{R}^N\right)$ is continuous provided from Lemma 4.1. Finding that from Lemma 4.1 (iii) there is $t_0\in(0,Z)$ making

$$\left|\gamma\left(t_0\right)\right|_2^2=m\ \text{ and }\ \left|\gamma\left(t_0\right)-h\right|_2\geq\delta,$$

From Lemma 4.1, we have item (ii) that we have be contradicted:

$$\begin{aligned}J_\lambda(h)&>J_\lambda\left(\gamma\left(t_0\right)\right)\\&=I\left(\gamma\left(t_0\right)\right)+\frac{\lambda}{2}\int_{\mathbb{R}^N}\left|\gamma\left(t_0\right)\right|^2\mathrm{d}x\\&\geq E_m+\frac{\lambda}{2}m.\end{aligned}$$

Having obtained that $m=|h|_2^2$, it is straightforward to see that $E_m=I(h)$ from (4.3).

5. Mountain Pass Structure

The next several lemmas are provided to show that I makes the $(PS)_c$ condition and the mountain pass structure achieved, leading to the energy functional I exists a nontrivial critical point u.

Lemma 5.1 There exists $\rho>0$ and $\alpha>0$, making $\inf_{\|u\|=\rho}I(u)\geq\alpha$.

Proof: According to (g_1) and (g_4), from each fixed $\varepsilon>0$, having $C_\varepsilon>0$, making for any $t\in\mathbb{R}$, we get $|G(t)|\leq C_\varepsilon|t|^p+\varepsilon|t|^2$.

Set $v\in H^1\left(\mathbb{R}^N\right)$ with $u\neq 0$. During the Sobolev Embedding Theorem, one has

$$\begin{aligned}I(v)&=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2-\frac{\mu}{|x|^2}v^2\right)\mathrm{d}x-\int_{\mathbb{R}^N}G(v)\mathrm{d}x\\&\geq\frac{1}{2}\|v\|^2-\int_{\mathbb{R}^N}\left(\varepsilon|v|^2+C_\varepsilon|v|^p\right)\mathrm{d}x\\&\geq\frac{1}{2}\|v\|^2-\varepsilon C_1\|v\|^2-C_\varepsilon C_2\|v\|^p,\end{aligned}$$

when choosing $\varepsilon > 0$ small sufficiently, we obtain

$$\begin{aligned} I(v) &\geq \frac{1}{4}\|v\|^2 - C_\varepsilon C_2 \|v\|^p \\ &= \frac{1}{4}\rho^2 - C_\varepsilon C_2 \rho^p = \alpha > 0, \end{aligned}$$

with ρ is small enough.

Lemma 5.2 Having $e \in H^1(\mathbb{R}^N)$, makes $I(e) < 0$.

Proof: When each $T > 0$, due to the continuity of u and the condition (g_6), having $C_3 > 0$ make for each $s \in \mathbb{R}$, we have $G(s) \geq Ts^4 - C_3$. Thus,

$$\begin{aligned} I(su) &= \frac{s^2}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 - \frac{\mu}{|x|^2}u^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} G(su)\,\mathrm{d}x \\ &\leq \frac{s^2}{2}\|u\|^2 - \int_{\mathbb{R}^N} T(su)^4\,\mathrm{d}x - C_3 \\ &= \frac{t^2}{2}\|u\|^2 - Ts^4\int_{\mathbb{R}^N} u^4\,\mathrm{d}x + C_4. \end{aligned}$$

It is evident that as $s \to \infty$, $I(su) \to -\infty$. Take $e = s_1 u$, where s_1 is enough large, we get $I(e) < 0$.

Lemma 5.1 and Lemma 5.2 have that I gets the mountain pass structure. To obtain the result, we should the next proof.

Lemma 5.3 The $(PS)_c$ condition of $I(u)$ is satisfied.

Proof: Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a $(PS)_c$ sequence of I, thus, $I(u_n) \to c$ and $I'(u_n) \to 0$, when $n \to \infty$. Nowaday, in order to prove that $\{u_n\}$ is a bounded sequence. Let $\{u_n\}$ is a unbounded sequence; without loss of generality, suppose $\|u_n\| \to \infty$, and by each $u, v \in H^1(\mathbb{R}^N)$,

$$\langle I'(v), u\rangle = \int_{\mathbb{R}^N}\left(\nabla v \nabla u - \frac{\mu}{|x|^2}vu\right)\mathrm{d}x - \int_{\mathbb{R}^N} g(v)v\,\mathrm{d}x. \tag{5.1}$$

Thus,

$$I(v_n) - \frac{1}{4}\langle I'(v_n), v_n\rangle = \frac{1}{4}\|v_n\|^2 + \int_{\mathbb{R}^N}\left(\frac{1}{4}g(v)v - G(v)\right)\mathrm{d}x.$$

For (g_5), having $C_5 > 0$ and $R > 0$ when $|t| > R$, $g(t)t - 4G(t) \geq C_1|t|^{2+\sigma} - C_2 t^2 \geq C_5$. Thus, one has

$$\frac{I(v_n) - \frac{1}{4}\langle I'(v_n), v_n\rangle}{\|v_n\|^2} = \frac{1}{4} + \frac{1}{4}\int_{\mathbb{R}^N}\frac{G(v_n)v_n - 4G(v_n)}{\|v_n\|^2}\mathrm{d}x \geq \frac{1}{4} + \frac{1}{4}\int_{\mathbb{R}^N}\frac{C_5}{\|v_n\|^2}\mathrm{d}x. \tag{5.2}$$

Furthermore, by assumption, we have $\dfrac{I(v_n) - \frac{1}{4}\langle I'(v_n), v_n\rangle}{\|v_n\|^2} \to 0$.

According to (5.2), $0 \geq \frac{1}{4}$, which is a contradiction. Therefore, $\{u_n\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$ and then one may assume that up to a subsequence $\{v_n\}$ (still denoted by $\{v_n\}$) satisfies $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and

$v_n \to v$ in $L^p\left(\mathbb{R}^N\right)$ for $1 \le p < 2^*$. Moreover, $v_n(x) \to v(x)$ a.e. on $\mathbb{R}^N$. By (g_4), Hölder's inequality and the properties of the Nemytskii operator, we obtain

$$\left|\int_{\mathbb{R}^N} \left(f(u_n) - f(u)\right)(u_n - u)\mathrm{d}x\right| \le \left|f(u_n) - f(u)\right|_{L^{p/(p-1)}\left(\mathbb{R}^N\right)} \left|u_n - u\right|_{L^p\left(\mathbb{R}^N\right)} \to 0. \quad (5.3)$$

Thus, according to (5.1) and (5.3), one has

$$\begin{aligned}\left\langle I'(v_n) - I'(v), v_n - v\right\rangle &= \left\langle I'(v_n), v_n - v\right\rangle - \left\langle I'(v), v_n - v\right\rangle \\ &= v_n - v^2 - \int_{\mathbb{R}^N} \left(g(v_n) - g(v)\right)(v_n - v)\mathrm{d}x \\ &= \left\|v_n - v\right\|^2 + o(1).\end{aligned}$$

Furthermore, by $I'(v_n) \to 0$ and the definition of weak convergence,

$$\left\langle I'(v_n) - I'(v), v_n - v\right\rangle \to 0, \text{ as } n \to \infty,$$

hence $\left\|v_n - v\right\| \to 0$. The lemma 5.3 is proved.

Theorem 1.5 of Proof

Due to Lemma 5.1 and Lemma 5.2, it is demonstrated that the energy functional I satisfied the mountain pass structure in $H^1\left(\mathbb{R}^N\right)$, Consequently, having a $(PS)_c$ sequence $\{u_n\}$ of I in $H^1\left(\mathbb{R}^N\right)$. Moreover, from Lemma 5.3, I makes the $(PS)_c$ condition. Because of Theorem 1.4 and $c > 0$, I makes a nontrivial critical point u, i.e., the functional I has a nontrivial solution u.

Conflicts of Interest

The author declares no conflicts of interest.

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